2.3 – Modeling with Exponential Functions

1 Introduction

Many real-world phenomena change rapidly and proportionally over time – population growth, compounding investments, and radioactive material decay as examples. These situations are often modeled using exponential functions.

Exponential functions take the general form

$$y = a \cdot b^x + d$$

where a, b, and d are constants, and x is the explanatory variable. The parameter a influences the initial value and vertical stretch, b determines the growth or decay rate (general shape), and d serves as a vertical shift.

The most basic form of the exponential function is $y = b^x$, where b is the constant base raised to a variable power x. Do not confuse this with the power function, where the variable base is raised to a constant power. **The exponential function has the independent variable in the exponent**, leading to unique behaviors like rapid growth or slow decay, depending on the value of b [1].

In this course we will also use exponential functions of the following form where r is the growth or decay rate of the function.

$$y = a(1 + r)^x + d$$

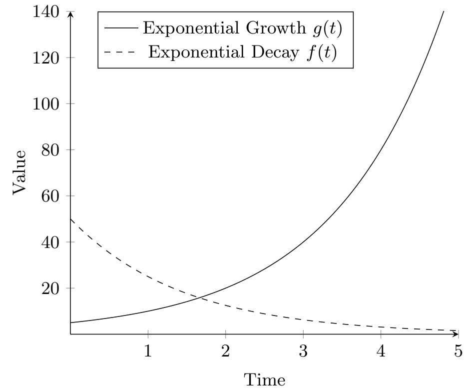


Figure 1: Exponential Growth and Decay: Examples include bacteria growth $g(t) = 5 \cdot 2^t$ (growth, $b > 1$) and radioactive decay $f(t) = 50 \cdot 0.5^t$ (decay, $0 < b < 1$).

Exponential Behavior Summary

- If $a > 0$, the graph of the function lies above the horizontal asymptote $y = d$.
- If $a < 0$, the graph of the function lies below the horizontal asymptote $y = d$.
- If $b > 1$, the function will diverge from the horizontal asymptote $y = d$ as x increases.
- If $0 < b < 1$, the function will move towards the horizontal asymptote $y = d$ as x increases.
- If $b = 1$, the function will remain a constant distance from the horizontal asymptote $y = d$ as x increases [1].

2 Characteristics

2.1 Growth vs. Decay

Exponential functions are characterized by rapid growth when the base $b > 1$ or rapid decay when $0 < b < 1$. They also feature a horizontal asymptote, typically at $y = d$. Note: when $b < 0$, the function is undefined for most values of x.

When using exponential functions of the form $y = a(1 + r)^x + d$, the function is growing when $r > 0$, and decaying when $r < 0$.

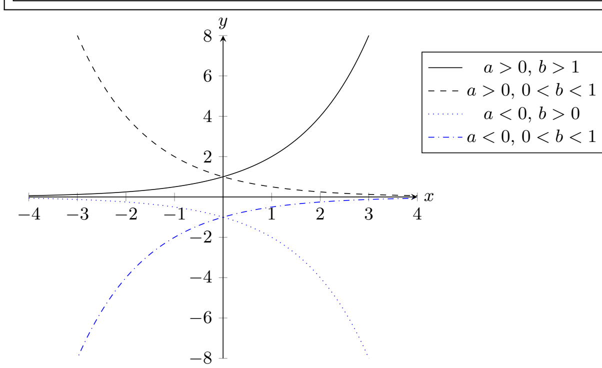


Figure 2: Cases of Exponential Functions: Demonstrating the effects of different a and b parameter combinations on graph behavior.

2.2 Applications

Exponential functions are suitable for scenarios in which changes occur proportionally to the current value, such as population growth, radioactive decay, finance (compound interest), and cooling of objects.

3 Estimating Parameters

The parameters a, b, and d of the generalized exponential function $f(x) = a \cdot b^x + d$ can be estimated through a combination of visual inspection and mathematical computation, similar to the approach used for linear functions. We will explore both a first principles and an empirical approach.

3.1 First Principles Approach

For an exponential model we must ask ourselves the following questions.

- What is the value of the dependent variable when the independent variable is 0? (e.g. what is the population at time 0?) This gives us $a + d$.
- Do we expect the dependent variable to grow or decay over time? ($r > 0$ or $r < 0$, $b > 1$ or $0 < b < 1$)
- By what rate do we expect the dependent variable to grow or decay? This gives us the value of the growth or decay rate, r.
- Is there some value toward which we expect the dependent variable to converge, or away from which we expect the dependent variable to diverge? These horizontal asymptotes give us the value of d.

3.2 Empirical Approach Method 1

- Using the plot of the data, determine the shape.
- Choose the appropriate exponential behavior (growth or decay).
- The y-value of the horizontal asymptote is a good initial estimate for the d parameter.
- Using two data points (x_1, y_1) and (x_2, y_2) that are representative of the general trend of the data, and the location of the horizontal asymptote as the d parameter estimate, solve the two equations:

$$y_1 = f(x_1) = a \cdot b^{x_1} + d$$

$$y_2 = f(x_2) = a \cdot b^{x_2} + d$$

for a and b. **Note:** The b parameter must be greater than zero.

3.3 Empirical Approach Method 2

- Using the plot of the data, determine the shape.
- Choose the appropriate exponential behavior.
- If the value of the horizontal asymptote cannot be estimated, choose three representative data points and solve the following system of equations:

$$f(x_1) = a \cdot b^{x_1} + d$$

$$f(x_2) = a \cdot b^{x_2} + d$$

$$f(x_3) = a \cdot b^{x_3} + d$$

for a, b, and d. **Note:** The b parameter must be greater than zero.

3.4 Empirical Approach Method 3

We will often use trendlines in Excel to build predictive models:

- Using the plot of the data, determine the shape
- Choose the appropriate exponential behavior.
- Excel gives the trendline in the form $y = ae^{bx}$. This is equivalent to the form $y = a(1 + r)^x$. This may not be immediately obvious, so we will walk through the following example to understand Excel's exponential trendline: say we estimate our model to be $y = 4(1 + 0.5)^x$ using a first principles approach, but Excel reports this in the form $y = 4e^{0.4055x}$. We can simply use our exponent rules to rewrite the Excel equation as $y = 4(e^{0.4055})^x$ and find the quantity $e^{0.4055}$ to be reasonably close to our base growth rate (r) of 1.5.

Problem 2.3.1: Worked Example Problem

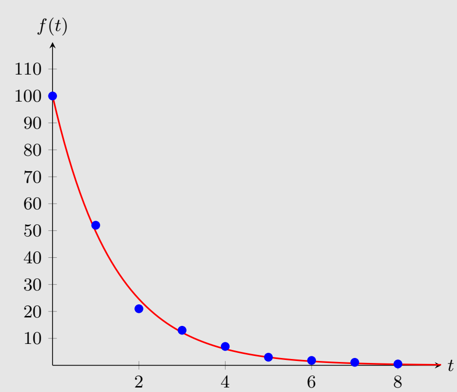
Suppose we have a radioactive substance and we are trying to predict the amount of substance remaining after a certain number of days. Each day, for the last 10 days, we measured the number of grams left, see Table 1. Model the grams of substance remaining as a function of days.

Table 1: Measurements over time

t	0	1	2	3	4	5	6	7	8
$f(t)$	100	52	21	13	7	3	1.8	1.1	0.5

Transform the Problem.

- Define variables. Let t be the independent variable representing the number of days since the first measurement and let $f(t)$ be the dependent variable representing the amount of substance remaining in grams.
- Assumption. The substance decreases by a fixed percentage every day. This is necessary to model the data using an exponential model and reasonable because radioactive material normally follows this pattern.
- Using the plotted data in the figure below, it appears that an exponential function with $a > 0$ and $0 < b < 1$ would work the best.



- It appears that the data is getting closer to 0 as the number of days increases, so we can assume the horizontal asymptote is at 0, and therefore we can assign $d = 0$.
- Since we can see the horizontal asymptote and assign a value to the d-parameter, we only need to select 2 points from the data set to solve for a and b using Method 2. We will use $(0, 100)$ and $(5, 3)$.
- Set up the system of equations:

$$100 = a \cdot b^0 + 0 \tag{1}$$
$$3 = a \cdot b^5 + 0 \tag{2}$$

- From Equation 1, we see that $100 = a$
- Then we can plug in the value for a into Equation 2 to solve for b.

$$3 = 100 \cdot b^5 \tag{3}$$
$$\frac{3}{100} = b^5 \tag{4}$$
$$(\frac{3}{100})^{\frac{1}{5}} = b \tag{5}$$
$$b = 0.496 \tag{6}$$

- Therefore, our estimated model is:

$$f(t) = 100 \cdot 0.496^t \tag{7}$$

- The calculated parameters support our initial expectation: $100 > 0$ and $0 < 0.496 < 1$. The estimated model is graphed in red in the figure above.

3.5 Model Assessment

We will assess our exponential models using SSE and R^2. We will use the R^2 value given in Excel for nonlinear models. Remember that we want to minimize SSE and that, generally, the closer R^2 is to 1, the better the fit.

4 Ethical Checklist

As we discussed in our introductory and linear readings, we must consider data and model validity, and clearly communicate our exponential models. Below are some questions to think through as you consider whether your exponential model is ethical.

- **Data validity**. Where did the data come from? How was it collected? How much did you need to clean the data? If you had to clean a lot of it, what are the possible implications about this data set?
- **Model validity**. Examine your modeling decisions.
  - Did you make modeling decisions while cleaning or exploring your data? How did you identify your explanatory and response variables, are they the best variables to choose? Does your model answer the question you are trying to find?
  - Is your predictive model accurate? How did you evaluate your model? How does your model perform on unseen data? Does your choice of exponential model make sense in the context of the problem?
- **Communication**. Is your visualization clear? Is it misleading, or does it convey an honest representation of the data? Did you thoroughly communicate each of your assumptions and modeling decisions? Did you communicate the limitations of your model?

References

[1] US Military Academy. *Modeling in a Real and Complex World*. West Point, New York: Department of Mathematical Sciences, 2022.