From the previous reading remember the basic formulation for a linear programming model (LP) [1],

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i = 1, 2, \ldots, m \\ & x_j \geq 0, \qquad j = 1, 2, \ldots, n \end{aligned}$$

which is composed of an *objective function* and *constraints*. While this formulation is built to handle any size problem, we are going to focus on problems with only two variables to generate a visual understanding of constraints and feasibility.

1 Formulating a Linear Program with Two Variables

Let's start with a problem to build through the reading.

Problem 3.2.1: The Wyndor Problem [2]

The Wyndor Glass Co. produces high-quality glass products, including windows and glass doors. There are three plants that produce these items for the company. Doors require one hour of production time at Plant 1 and three hours of production time at Plant 3. Windows require two hours each at Plants 2 and 3. Plant 1 has 4 hours available each week, Plant 2 has 12 hours available each week, and Plant 3 has 18 hours available each week. How many windows and doors should Wyndor Glass Co. produce each week?

Problem 1 does not have a clear objective statement, so we are going to ignore developing an objective function for now. However, other pertinent information is available that we can use to model or formulate our constraints. Our constraints follow the general form:

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j \geq b_i, & \qquad i = 1, 2, \ldots, m \\ x_j \geq 0, & \qquad j = 1, 2, \ldots, n \end{aligned}$$

Let's remember what these mean again and then apply the context of Problem 1 to them.

The variables for the problem, or the decision to be made, are represented by x_j. Within the context of this problem, we have two decisions to make: the number of doors to produce in a week and the number of windows to produce in a week. We can define our variables in the following way:

x_1 = the number of doors produced in one week by Wyndor Glass Co.

x_2 = the number of windows produced in one week by Wyndor Glass Co.

Next let's examine what our constraints need to be. We are given that there are three plants. Each product type has a required number of hours at each plant and each plant has a total number of hours available each week. We know from the previous readings that a_{ij} for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ describe how much of resource i is used by one unit of variable j [1]. In this context, the resource is time available at each plant, and the technological coefficients therefore represent the time each product requires at each plant. Our general definition for our technological coefficients can be written as:

a_{ij} = the amount of time in hours it takes at Plant i to produce one of each product j

where $i = 1, 2, 3$ and $j = 1, 2$, respectively. We can write each in turn using words:

a_{11} = the amount of time in hours it takes at Plant 1 to produce one door

a_{12} = the amount of time in hours it takes at Plant 1 to produce one window

a_{21} = the amount of time in hours it takes at Plant 2 to produce one door

a_{22} = the amount of time in hours it takes at Plant 2 to produce one window

a_{31} = the amount of time in hours it takes at Plant 3 to produce one door

a_{32} = the amount of time in hours it takes at Plant 3 to produce one window

The values b_i are the total amount of resource i available [1]. Again, if our resource is time, then

b_i = the total time in hours available at Plant i

where $i = 1, 2, 3$. We can also write out each b_i clearly in words:

b_1 = the total time in hours available at Plant 1

b_2 = the total time in hours available at Plant 2

b_3 = the total time in hours available at Plant 3

With variables and coefficients clearly defined, we can now express the constraints algebraically. These inequalities represent the time limitations at each plant. Also, we have to remember that we cannot make a negative number of doors or windows, so we need to include nonnegativity constraints, too. Note that these expressions complete the transform step of modeling the constraints. We define the general structure of the model symbolically, without yet substituting numerical values. Specific parameter values will be introduced in the solve step. The full algebraic constraints for this problem are:

$$\begin{array}{ccccccc} a_{11} & x_1 & + & a_{12} & x_2 & \leq & b_1 \\ a_{21} & x_1 & + & a_{22} & x_2 & \leq & b_2 \\ a_{31} & x_1 & + & a_{32} & x_2 & \leq & b_3 \\ & x_1 & & & & \geq & 0 \\ & & & & x_2 & \geq & 0 \end{array}$$

Note: While the general LP formulation often uses $\geq$ to represent constraints (e.g., minimum requirements), the Wyndor problem involves resource limitations; each plant has at most a certain number of hours available. Therefore, the constraints here are expressed with $\leq$ to indicate these upper bounds.

When we are ready to solve, we can *finally(!)* substitute actual values for our parameters into our constraints. Our formulation for the model constraints that we will use to solve the problem are:

$$\begin{array}{ccccccc} 1 & x_1 & + & 0 & x_2 & \leq & 4 \\ 0 & x_1 & + & 2 & x_2 & \leq & 12 \\ 3 & x_1 & + & 2 & x_2 & \leq & 18 \\ & x_1 & & & & \geq & 0 \\ & & & & x_2 & \geq & 0 \end{array}$$

These numerical constraints now define the specific limits on how many windows and doors Wyndor can produce given available resources. These effectively limit what is possible within the model. To analyze the model further, we must determine which values of the decision variables meet all of these constraints simultaneously.

2 Feasibility

In the modeling process, once the model has been formulated, we explore what values of the decision variables satisfy all constraints. This is part of the solve step, where we begin analyzing the behavior of the model to understand which solutions are possible.

A point at which all of the constraints are met simultaneously is called a *feasible solution* [2]. These solutions collectively form the *feasible region*, or the set of all possible points that satisfy every constraint.

> **Definition 3.2.1** (Feasible Region [3])
> The set of all points that satisfies all the LP's constraints.

2.1 Checking Feasibility

Before we look at how we can visualize a feasible region, let's think more about what it means to be a feasible solution. Let's say Wyndor Glass Co. decides to produce 2 doors and 4 windows every week. Is this a feasible solution? We have to check to see if any of our constraints are violated using $x_1 = 2$ and $x_2 = 4$.

$$\begin{array}{lcc} \text{Plant 1} & 1x_1 + 0x_2 & \leq 4 \\ & 1(2) + 0(4) & \leq 4 \\ & 2 \leq 4 & \checkmark \\ \\ \text{Plant 2} & 0x_1 + 2x_2 & \leq 12 \\ & 0(2) + 2(4) & \leq 12 \\ & 8 \leq 12 & \checkmark \\ \\ \text{Plant 3} & 3x_1 + 2x_2 & \leq 18 \\ & 3(2) + 2(4) & \leq 18 \\ & 14 \leq 18 & \checkmark \end{array}$$

We also check the nonnegativity constraints and see that $2 \geq 0$ and $4 \geq 0$. Since all constraints are satisfied, the solution $(x_1, x_2) = (2, 4)$ is a feasible solution! Problem 2 contains three more worked examples checking solutions for feasibility.

If a proposed solution violates even one constraint, including the nonnegativity constraints, it is considered infeasible.

Note: At this stage we are only checking whether a solution is valid, not whether it is the best solution. That comes after we've identified the full set of feasible solutions.

2.2 Visualizing the Feasible Region

We can visualize the feasible region, or set of all points that satisfy the LP's constraints for an LP with two variables by graphing the constraints and determining where they overlap. Let's continue to use the Problem 1 as an example.

Let's start by graphing the first constraint $x_1 \leq 4$ as shown in Figure 1. Remember when we graph inequalities, the boundary line itself is included, because the inequality is defined with a $\leq$ symbol. If we were graphing $<$, then the boundary line would not be included.

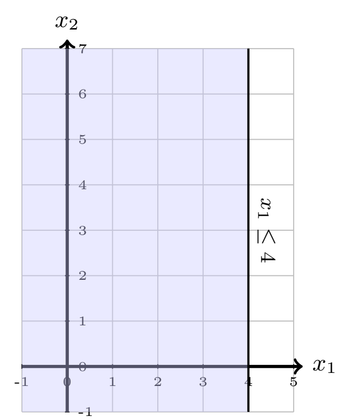

Figure 1: Graph of the first constraint, the inequality $x_1 \leq 4$.

We continue by graphing the remaining constraints to identify where all shaded regions overlap. From Figure 2 we can begin to identify this region. If we add in our nonnegativity constraints, which restricts us to the first quadrant of our graph, this overlapping region represents the set of all solutions that simultaneously satisfy all constraints, or our *feasible region*. The cleaned up graph depicting the feasible region is shown in Figure 3.

Problem 3.2.2: Feasible Solutions

Using the context provided in Problem 1, determine if the following solutions are feasible or not.

$$(x_1, x_2) = (3, 4), \quad (x_1, x_2) = (4, 6),$$
$$(x_1, x_2) = (-1, 2)$$

Solution:

Check $(x_1, x_2) = (3, 4)$:

$$3 \geq 0 \quad \checkmark$$
$$4 \geq 0 \quad \checkmark$$

$$\begin{aligned} \text{Plant 1} \quad 1x_1 + 0x_2 &\leq 4 \\ 1(3) + 0(4) &\leq 4 \\ 3 \leq 4 & \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Plant 2} \quad 0x_1 + 2x_2 &\leq 12 \\ 0(3) + 2(4) &\leq 12 \\ 8 \leq 12 & \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Plant 3} \quad 3x_1 + 2x_2 &\leq 18 \\ 3(3) + 2(4) &\leq 18 \\ 17 \leq 18 & \quad \checkmark \end{aligned}$$

$(x_1, x_2) = (3, 4)$ is a feasible solution.

Check $(x_1, x_2) = (4, 6)$:

$$4 \geq 0 \quad \checkmark$$
$$6 \geq 0 \quad \checkmark$$

$$\begin{aligned} \text{Plant 1} \quad 1x_1 + 0x_2 &\leq 4 \\ 1(4) + 0(6) &\leq 4 \\ 4 \leq 4 & \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Plant 2} \quad 0x_1 + 2x_2 &\leq 12 \\ 0(4) + 2(6) &\leq 12 \\ 12 \leq 12 & \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Plant 3} \quad 3x_1 + 2x_2 &\leq 18 \\ 3(4) + 2(6) &\leq 18 \\ 24 \nleq 18 & \quad \text{X} \end{aligned}$$

$(x_1, x_2) = (4, 6)$ is NOT a feasible solution.

Check $(x_1, x_2) = (-1, 2)$:

$$-1 \ngeq 0 \quad \text{X}$$
$$2 \geq 0 \quad \checkmark$$

$(x_1, x_2) = (-1, 2)$ is NOT a feasible solution.

If we pick any point within the shaded region, or on the boundary of our region, we are guaranteed to have a feasible solution. Let's demonstrate this by plotting the solutions we evaluated previously and in Problem 2 against the feasible region in Figure 4.

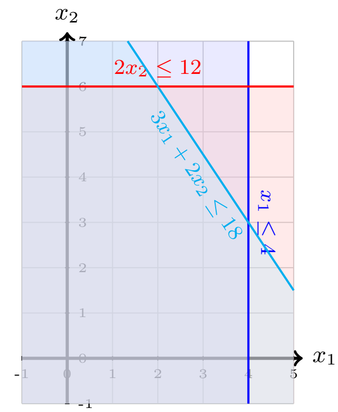

Figure 2: Graph of all constraints except for nonnegativity from Problem 1.

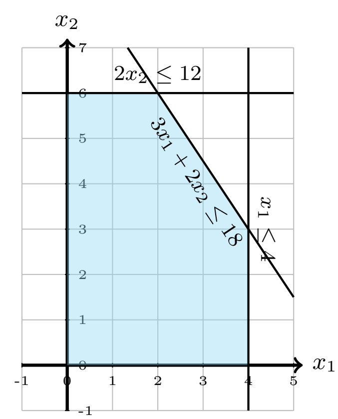

Figure 3: Graph of the feasible region for Problem 1.

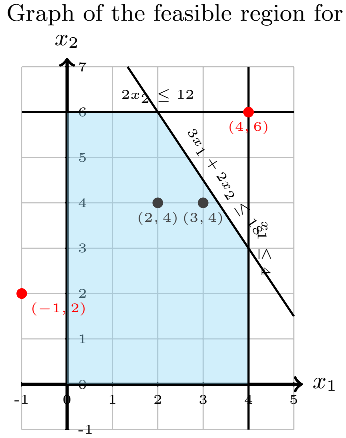

Figure 4: Possible solutions plotted against the feasible region. Note that the red points lie outside our feasible region and are, therefore, not feasible solutions.

References

[1] Mokhtar S. Bazaraa, John J. Jarvis, and Hanif D. Sherali. *Linear Programming and Network Flows.* Wiley, 2010.

[2] Frederick Hillier et al. *MA103 Mathematical Modeling: Introduction to Management, Science, & Business Analytics with Connect.* McGraw-Hill, 2024.

[3] Wayne Winston and Munirpallam Venkataramanan. *Introduction to Mathematical Programming.* Brooks/Cole, 2003.