1 Vector Basics

A vector is a mathematical construct used to represent quantities that have both magnitude and direction. Unlike scalars, which only have magnitude (e.g., mass or temperature), vectors describe quantities like force, velocity, and displacement. For example, the speed of a vehicle is a scalar, while velocity is a vector because it includes the direction of travel. Vectors are essential tools in physics, engineering, computer graphics, and other fields where direction and magnitude are important. Because vectors are defined by their magnitude and direction, not by their location, vectors can be freely translated to start from any point in space.

1.1 Component Form

In two dimensions, vectors are typically represented as an ordered set of numbers, such as $\vec{v} = \langle v_1, v_2 \rangle$, where v_1 and v_2 represent the components in the x- and y-directions, respectively. Vectors can also represent the position of a point relative to the origin, called a *position vector*.

Definition M.1.1 (Position Vector)
A position vector represents the location of a point relative to the origin. For example, the position vector of the point $A(3, 2)$ is $\vec{a} = \langle 3, 2 \rangle$.

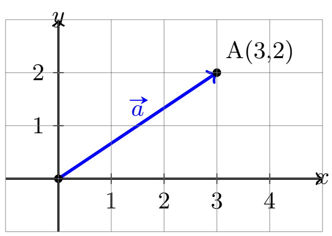

Figure 1: The point $A(3, 2)$ and its position vector $\vec{a} = \langle 3, 2 \rangle$. The tail of the vector starts at the origin, and the head terminates at point A.

Vectors in three dimensions allow us to represent quantities in 3D space using three coordinates: x, y, and z. For example, the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ has components representing displacements in the x-, y-, and z-directions. The three-dimensional system typically uses the *right-hand rule* to define the positive directions of the axes [1].

Definition M.1.2 (Right-Hand Rule)
If you point the index finger of your right hand in the direction of the positive x-axis, and curl your fingers toward the positive y-axis, your thumb points in the direction of the positive z-axis.

We define the size or *dimension* of a vector based on the number of components it contains. So a vector with two components is a 2-dimensional vector; a vector with three components is a 3-dimensional vector, etc. Generally, in this course we will use 2D and 3D vectors, but know that vectors of any size do exist. The dimension of the vector determines the operations we are allowed to conduct. We will discuss this later on.

1.2 Magnitude of a Vector

Magnitude is another name for the length of the vector [1]. In two or three dimensions, we could plot and measure the lengths. However, the magnitude of a vector can be calculated for a vector with any number of dimensions.

Definition M.1.3 (Magnitude)
The magnitude of an n-dimensional vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is defined as:

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

The terms **magnitude**, and **norm** are all synonymous.

It might seem strange at first to use absolute value bars to also represent the length of a vector. However, they are similar concepts. The absolute value of zero is zero, and the absolute value of any other number is positive. Similarly, the length of the zero vector is zero, and the length of any other vector is positive. The absolute value of a number represents its distance on the number line from zero, while the length of a position vector represents the distance from the origin to its head. The absolute value of a number is its magnitude, ignoring its sign; whereas the length of a vector is its magnitude, ignoring its direction.

Definition M.1.4 (Zero Vector)
A vector where every component is 0.

$$\vec{0} = \langle 0, 0, \ldots, 0 \rangle$$

1.3 Unit Vector

A unit vector is a vector that has a length of one unit. We will annotate it with a vector hat such as $\hat{u}$, instead of the typical vector arrow. We can find the unit vector $\hat{u}$ in the same direction of any vector $\vec{u}$ by dividing the vector by its magnitude. This is the same as multiplying each component of the vector by one over the vector's magnitude.

Definition M.1.5 (Unit Vector)
The unit vector $\hat{u}$ of a vector $\vec{u}$ is defined as:

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$$

1.4 Standard Basis Vectors

We can express any vector as a linear combination of the standard basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$, which point in the positive x-, y-, and z-directions, respectively, and have a length of one unit [1]:

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$
$$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

For example, the vector $\vec{a} = \langle 2, 3, 4 \rangle$ can be written as:

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

This shows that the vector $\vec{a}$ is the resultant of a displacement two units in the direction of the positive x-axis, three units in the direction of the positive y-axis, and four units in the direction of the positive z-axis, shown in Figure 2. These displacements are expressed using scalar multiples of **unit vectors**, in each of these directions.

Even though we haven't formally defined it yet, writing $2\hat{i}$ is an example of *scalar multiplication*; we're scaling the unit vector $\hat{i}$ by a factor of 2. This operation stretches or shrinks a vector's length, and (for now) you can think of it as preserving the direction when the scalar is positive. We'll define scalar multiplication more precisely, including what happens when we multiply by a negative scalar, in Section 2.1, but for now, you can think of it as a way to control how far you move in each direction.

Definition M.1.6 (Standard Basis Vectors)
The standard basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors that point in the positive x-, y-, and z-directions, respectively. Their components are:

$$\hat{i} = \langle 1, 0, 0 \rangle, \quad \hat{j} = \langle 0, 1, 0 \rangle, \quad \hat{k} = \langle 0, 0, 1 \rangle$$

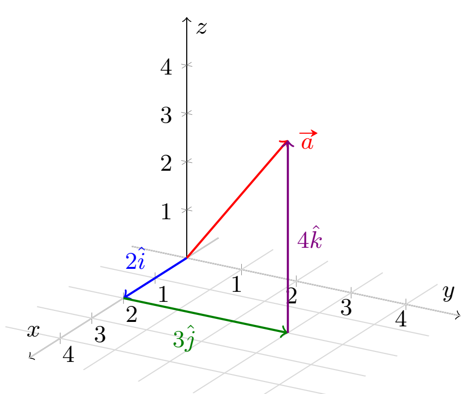

Figure 2: Vector $\vec{a} = \langle 2, 3, 4 \rangle$ and its components $2\hat{i}, 3\hat{j}, 4\hat{k}$, with a grid on the x-y plane.

2 Vector Operations

Vectors can be manipulated using operations such as scalar multiplication, addition, and subtraction. In the cases of addition and subtraction the operation can be done only if the two vectors being added or subtracted have the same dimension, or the same number of components. These operations allow us to combine or modify vectors to model real-world scenarios.

2.1 Scalar Multiplication

A vector can be scaled by multiplying it by a scalar (a real number).

Definition M.1.7 (Scalar Multiplication)
Scalar multiplication scales a vector by a real number c. It changes the magnitude of the vector and may reverse its direction if $c < 0$. Given $\vec{u} = \langle u_1, u_2 \rangle$, the scalar multiplication $c\vec{u}$ is:

$$c\vec{u} = c \cdot \langle u_1, u_2 \rangle = \langle cu_1, cu_2 \rangle$$

Effects of scalar multiplication:

- $1 < |c|$: Increases magnitude.
- $0 < |c| < 1$: Reduces magnitude.
- $c < 0$: Reverses direction and scales by $|c|$.

In Figure 3, we can see scalar multiples of: $\vec{u} = \langle 2, 1 \rangle$

$$2\vec{u} = \langle 4, 2 \rangle \qquad -0.5\vec{u} = \langle -1, -0.5 \rangle$$

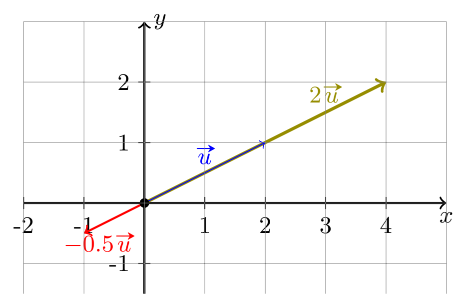

Figure 3: Graphical scalar multiplication.
Note: The three vectors depicted start at the origin.

2.2 Vector Addition

The addition operation is by component. For two vectors $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, the sum is:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$$

Problem M.1.1: Vector Addition

If $\vec{u} = \langle 2, 1\rangle$ and $\vec{v} = \langle 1, 2\rangle$ find $\vec{u} + \vec{v}$.

Solution:

$$\begin{aligned}\vec{u} + \vec{v} &= \langle 2, 1\rangle + \langle 1, 2\rangle \\ &= \langle 2, 1\rangle + \langle 1, 2\rangle \\ &= \langle 3, 3\rangle\end{aligned}$$

Graphically, vector addition can be represented by placing the tail of $\vec{v}$ at the head of $\vec{u}$. The resulting vector from the tail of $\vec{u}$ to the head of $\vec{v}$ is the vector $\vec{u} + \vec{v}$ as seen in Figure 4.

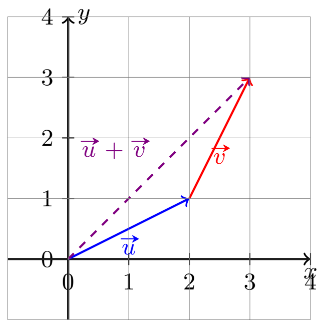

Figure 4: Graphical vector addition for Problem 1.

2.3 Vector Subtraction

The difference between two vectors $\vec{u} - \vec{v}$ is:

$$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2\rangle$$

Graphically, subtracting a vector is equivalent to adding its negative: $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$. Reverse $\vec{v}$ by scalar multiplication to obtain $-\vec{v}$ and then follow the graphical steps for vector addition.

Problem M.1.2: Vector Subtraction

If $\vec{u} = \langle 2, 1\rangle$ and $\vec{v} = \langle 1, 2\rangle$ find $\vec{u} - \vec{v}$.

Solution:

$$\begin{aligned}\vec{u} - \vec{v} &= \langle 2, 1\rangle - \langle 1, 2\rangle \\ &= \langle 2, 1\rangle + \langle -1, -2\rangle \\ &= \langle 2 - 1, 1 - 2\rangle \\ &= \langle 1, -1\rangle\end{aligned}$$

2.4 Displacement Vector

The displacement vector represents the change in position from one point to another. Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the displacement vector from A to B is:

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1\rangle$$

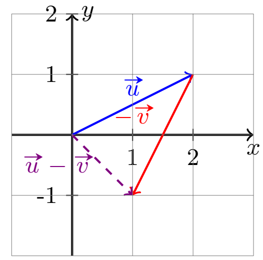

Figure 5: Graphical representation of vector subtraction as addition with scalar multiplication by -1 as shown in Problem 2.

Let $\vec{a} = \langle x_1, y_1\rangle$ and $\vec{b} = \langle x_2, y_2\rangle$ be the position vectors for points A and B. Then:

$$\overrightarrow{AB} = \vec{b} - \vec{a}$$

Graphically, this is equivalent to $\vec{a} + \overrightarrow{AB} = \vec{b}$ as shown in Figure 6.

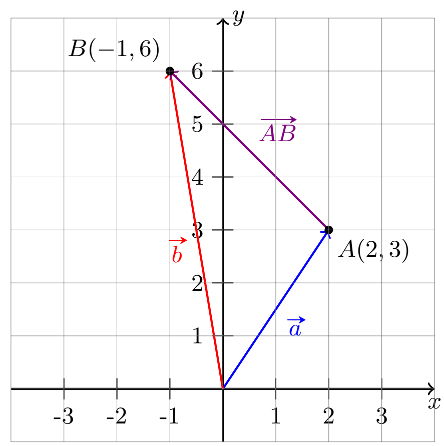

Figure 6: Graphical depiction of the displacement vector $\overrightarrow{AB}$ as calculated in Problem 3.

Problem M.1.3: Displacement Vector

Given points $A(2, 3)$ and point $B(-1, 6)$ find displacement vector $\overrightarrow{AB}$.

Solution:

Displacement vector $\overrightarrow{AB}$ means the vector starts at A and terminates at B.

$$\vec{a} = \langle 2, 3\rangle \qquad \vec{b} = \langle -1, 6\rangle$$

$$\begin{aligned}\overrightarrow{AB} &= \vec{b} - \vec{a} \\ &= \langle -1, 6\rangle - \langle 2, 3\rangle \\ &= \langle -1, 6\rangle + \langle -2, -3\rangle \\ &= \langle -1 - 2, 6 - 3\rangle \\ &= \langle -3, 3\rangle\end{aligned}$$

3 Dot Product

So far, we have discussed the basic vector operations of addition, subtraction, and scalar multiplication. Now, we explore another useful vector operation, called the dot product. We will analyze the dot product algebraically and conduct a geometric analysis of the dot product. Understanding the dot product is essential to our use of matrix algebra in later lessons.

Definition M.1.8 (**Dot Product**[1])
The dot product, or scalar product, given two vectors $\vec{w} = \langle w_1, w_2, ...\rangle$ and $\vec{v} = \langle v_1, v_2, ...\rangle$ is

$$\vec{w} \cdot \vec{v} = w_1 v_1 + w_2 v_2 + \cdots + w_n v_n = \sum_{i=1}^{n} w_i v_i$$

IMPORTANT: The dot product can only be performed between two vectors that have the same dimension (same number of components). If one vector has more dimensions or components than another vector then the dot product cannot be computed between the two vectors.

Problem M.1.4: Dot Product - Algebraic

Given that $\vec{w} = \langle 3, 4\rangle$ and $\vec{v} = \langle 5, 12\rangle$, find $\vec{w} \cdot \vec{v}$.

Solution:

$$\begin{aligned} \vec{w} \cdot \vec{v} &= \langle 3, 4\rangle \cdot \langle 5, 12\rangle \\ &= 3(5) + 4(12) \\ &= 15 + 48 \\ &= 63 \end{aligned}$$

Using geometric analysis we are able to come up with some other useful definitions and theorems using dot product.

Definition M.1.9 (**Dot Product - Geometric**[1])
The dot product of two n-dimensional vectors $\vec{w}$ and $\vec{v}$, denoted $\vec{w} \cdot \vec{v}$, is given by:

$$\vec{w} \cdot \vec{v} = |\vec{w}||\vec{v}| \cos(\theta)$$

where θ is the angle between the two vectors when placed tail to tail.

Problem M.1.5: Dot Product - Geometric

Given $|\vec{w}| = 5$, $|\vec{v}| = 13$, and the angle between them is 0.249 radians, find $\vec{w} \cdot \vec{v}$.

Solution:

$$\begin{aligned} \vec{w} \cdot \vec{v} &= |\vec{w}||\vec{v}| \cos(\theta) \\ &= 5(13) \cos(0.249) \\ &= 65(0.969) \\ &\approx 63.0 \end{aligned}$$

We can rewrite the geometric definition of the dot product solve for the angle between the two vectors.

Definition M.1.10 (**Angle Between Two Vectors** [1])
If θ is the angle between two nonzero vectors $\vec{w}$ and $\vec{v}$ then,

$$\theta = \cos^{-1}\left(\frac{\vec{w} \cdot \vec{v}}{|\vec{w}||\vec{v}|}\right),$$

where $0 \leq \theta \leq \pi$

Using the geometric definition we can use the dot product to quickly determine whether two vectors are parallel or orthogonal (perpendicular).

Theorem 1
Two non-zero vectors $\vec{u}$ and $\vec{v}$ are orthogonal, or perpendicular, written as $\vec{u} \perp \vec{v}$, if and only if [1]

$$\vec{u} \cdot \vec{v} = 0$$

Because,

$$\begin{aligned} |\vec{u}||\vec{v}| \cos\left(\frac{\pi}{2}\right) &= 0 \\ |\vec{u}||\vec{v}|(0) &= 0 \end{aligned}$$

Consider the two vectors plotted in Figure 7.

$$\vec{u} = \langle 1, 2\rangle, \qquad \vec{v} = \langle -2, 1\rangle$$

Their dot product is:

$$\vec{u} \cdot \vec{v} = 1(-2) + 2(1) = -2 + 2 = 0$$

Since $\vec{u} \cdot \vec{v} = 0$, the vectors $\vec{u}$ and $\vec{v}$ are orthogonal.

Theorem 2
Two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel, written as $\vec{u} \parallel \vec{v}$, if and only if the angle between them is either 0 or π and will have the following property [1]:

$$\vec{u} = c\vec{v}, \quad \text{for } c \neq 0$$

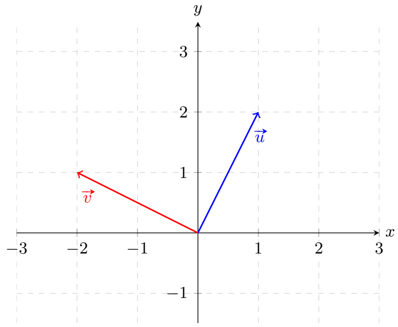

Figure 7: Plot of vectors $\vec{u}$ and $\vec{v}$ on the x-y plane.

Consider the vectors:

$$\vec{a} = \langle 3, 6 \rangle, \quad \vec{b} = \langle 1, 2 \rangle.$$

We can see that:

$$\vec{a} = 3\vec{b}, \quad \text{where } c = 3 \text{ and } c \neq 0.$$

Thus, $\vec{a}$ is a scalar multiple of $\vec{b}$ and therefore parallel to each other, as shown in Figure 8.

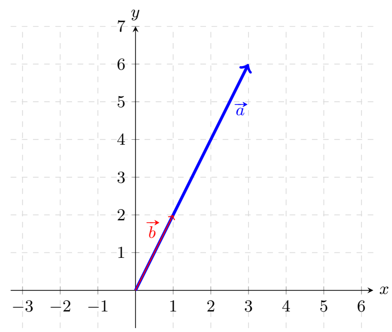

Figure 8: Plot of vectors $\vec{a}$ and $\vec{b}$ on the x-y plane.

4 Modeling with Vectors

In real-world applications, we often know the magnitude and direction of a vector, such as the force along a rope or the velocity of a moving object. To use vectors in a model, we typically need to convert that information into component form. This allows us to combine vectors, calculate net effects, and solve equations more easily.

4.1 Components from Magnitude and Direction

We can find the components of a vector and write a vector in component notation knowing the direction and magnitude of a vector. For example, if we know a vector $\vec{v}$ has a magnitude of 8, and we know it makes an angle of $\frac{\pi}{6}$ with the positive x-axis, we can use trigonometry to calculate the x and y components of the vector. The vector is plotted in Figure 9.

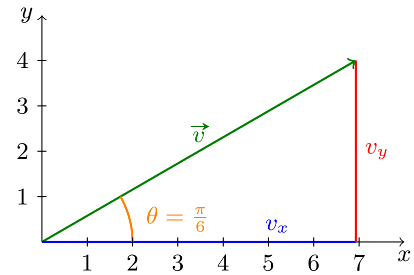

Figure 9: The vector $\vec{v}$ in a rectangular coordinate system.

Using trigonometry, the vector $\vec{v}$ is decomposed into its components:

the x-component is

$$\begin{aligned} v_x &= |\vec{v}|\cos\theta \\ &= 8\cos\left(\frac{\pi}{6}\right) \\ &= 4\sqrt{3} \\ &\approx 6.93 \end{aligned}$$

the y-component is

$$\begin{aligned} v_y &= |\vec{v}|\sin\theta \\ &= 8\sin\left(\frac{\pi}{6}\right) \\ &= 4 \end{aligned}$$

resulting in a component notation of $\vec{v} = \langle v_x, v_y \rangle = \langle 6.93, 4 \rangle$. This is generalized and captured in the definition below.

Definition M.1.11 (Calculating Vector Components [1])
The components of a vector $\vec{v}$ in the xy-plane, which has a magnitude $|\vec{v}|$ and makes an angle θ measured counter-clockwise from the x-axis, are given by:

$$v_x = |\vec{v}|\cos\theta \quad \text{and} \quad v_y = |\vec{v}|\sin\theta$$

4.2 Modeling Forces on a Stationary Mass

A stationary mass suspended by two strings, as shown in Figure 10, experiences no net force, meaning the forces from the strings ($\vec{F}_1$ and $\vec{F}_2$) must balance the gravitational force ($\vec{F}_g$) of 98 N downward. This means that $\vec{F}_g = \vec{F}_1 + \vec{F}_2$. Using a coordinate system where the x-axis is parallel to the surface, the angles of $\vec{F}_1$ and $\vec{F}_2$ relative to the x-axis are $\frac{5\pi}{6}$ and $\frac{\pi}{3}$, respectively, calculate the magnitudes of the forces required to keep the mass balanced.
The gravitational force can be written as $\vec{F}_g = \langle 0, -98 \rangle$. Decomposing the forces $\vec{F}_1$ and $\vec{F}_2$ into their components

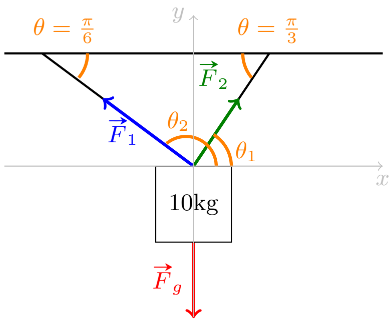

Figure 10: A stationary mass suspended by two strings.

using Definition 4.1, we have:

$$\vec{F}_1 = \langle |\vec{F}_1| \cos\left(\frac{5\pi}{6}\right), |\vec{F}_1| \sin\left(\frac{5\pi}{6}\right)\rangle$$
$$\vec{F}_2 = \langle |\vec{F}_2| \cos\left(\frac{\pi}{3}\right), |\vec{F}_2| \sin\left(\frac{\pi}{3}\right)\rangle$$

For equilibrium, the sum of the x- and y-components must equal zero:

The x-components of force:

$$|\vec{F}_1| \cos\left(\frac{5\pi}{6}\right) + |\vec{F}_2| \cos\left(\frac{\pi}{3}\right) = 0$$

The y-components of force:

$$|\vec{F}_1| \sin\left(\frac{5\pi}{6}\right) + |\vec{F}_2| \sin\left(\frac{\pi}{3}\right) - 98 = 0$$

Solving this system of equations yields:

$$|\vec{F}_1| = 49\,\text{N} \text{ and } |\vec{F}_2| = 84.9\,\text{N}$$

This approach, breaking vectors into components and using vector equations, is essential to solving modeling problems involving forces, motion, and direction. It's how we translate real-world quantities into mathematics we can work with.

References

[1] US Military Academy. *Modeling in a Real and Complex World*. West Point, New York: Department of Mathematical Sciences, 2022.