M.2 – Modeling with Matrices

1 Matrix Basics

Matrices are fundamental elements in linear algebra and essential for various mathematical and scientific computations. A matrix is an $n \times k$ array of numbers, where n represents the number of rows and k the number of columns. In this reading we will explore matrix properties and operations and their uses. In mathematical notation, a matrix A of size $n \times k$ is represented as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix}$$

Each element a_{ij} is located in the i-th row and j-th column. Matrix operations are generally classified into addition, subtraction, scalar multiplication, matrix multiplication, and transposition. Each of these operations has specific rules and requirements to be defined.

For example, this matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is a 2×2 matrix because there are two rows and two columns. A matrix with the same number of rows as columns is called a **square matrix**.

The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3×2 matrix.

A vector can also be written as either a $1 \times k$ or a $k \times 1$ matrix. For example, $\vec{v} = \langle 2, 1 \rangle$, can be written as the **row vector**, $\vec{v} = \begin{bmatrix} 2 & 1 \end{bmatrix}$, or the **column vector**, $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. We will often write $\vec{v}$ as a column vector to solve systems of equations.

2 Matrix Operations

2.1 Matrix Addition and Subtraction

Two matrices A and B can be added or subtracted if they have the same dimensions. The result of addition $A + B$ (or subtraction $A - B$) is a matrix where each element is the sum (or difference) of the corresponding elements in A and B.

Undefined Case: If matrices do not share identical dimensions, their sum or difference is undefined.

For example, if matrices A and B are both of size 2×3, we can add or subtract them. But if matrix A is of size 2×3 and matrix C is of size 2×4, addition or subtraction of these matrices is undefined.

Problem M.2.1: Addition and Subtraction

Given matrices $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 7 & 6 \end{bmatrix}$, perform the following:

$$A + B = \begin{bmatrix} 2+1 & 3+0 \\ 4+7 & 5+6 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 11 & 11 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-1 & 3-0 \\ 4-7 & 5-6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & -1 \end{bmatrix}$$

2.2 Scalar Multiplication

Multiplying a matrix A by a scalar c involves multiplying every element in A by c. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $cA = \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ c \cdot a_{21} & c \cdot a_{22} \end{bmatrix}$. Scalar multiplication works regardless of the matrix size.

Problem M.2.2: Scalar Multiplication

Given matrices $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and scalar $c = 2$, perform the following:

$$cA = 2 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$$

2.3 Matrix Multiplication

For two matrices A (of size $m \times n$) and B (of size $n \times p$), the product AB is defined and results in an $m \times p$ matrix. Each element in the resulting matrix is calculated by taking the dot product of the rows of A with the columns of B. Note that for matrix multiplication to be defined, the number of columns in A must equal the number of rows in B.

$$\underset{(m \times n)}{A} \; \underset{(n \times p)}{B} = \underset{(m \times p)}{C}$$

$$AB = \underset{(3 \times 2)}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}} \underset{(2 \times 3)}{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}}$$

$$AB = \underset{(3 \times 3)}{\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{bmatrix}}$$

Undefined Case: Matrix multiplication is undefined if the number of columns in A does not equal the number of rows in B.

$$\underset{(m \times n)}{A} \underset{(p \times k)}{B} \text{ is undefined,} \qquad \text{when } n \neq p$$

Note: Matrix multiplication is non-commutative. Meaning $AB \neq BA$, except in a few cases. Therefore we must distinguish between *left-multiplication* and *right-multiplication* when multiplying matrices.

Problem M.2.3: Matrix Multiplication

Given matrices $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 7 & 6 \end{bmatrix}$, perform the following:

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 7 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} (2 \cdot 1 + 3 \cdot 7) & (2 \cdot 0 + 3 \cdot 6) \\ (4 \cdot 1 + 5 \cdot 7) & (4 \cdot 0 + 5 \cdot 6) \end{bmatrix}$$

$$AB = \begin{bmatrix} 23 & 18 \\ 39 & 30 \end{bmatrix}$$

2.4 Transpose of a Matrix

The transpose of a matrix A, denoted A^T, is obtained by writing the rows of A as the columns of A^T. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$.

Problem M.2.4: Transpose

Given matrices $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 7 & 6 \end{bmatrix}$, perform the following:

$$A^T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 0 \\ 7 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 7 \\ 0 & 6 \end{bmatrix}$$

2.5 Vector Multiplication

When a $n \times k$ matrix multiplies a $k \times 1$ vector, the result is another vector of dimension $n \times 1$. For example, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then

$$A\vec{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix}$$

Problem M.2.5: Vector Multiplication

Given matrices $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, perform the following:

$$A\vec{v} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot 2 \\ 4 \cdot 1 + 5 \cdot 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

$$\vec{v}A = \underset{(2\times 1)}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{(2\times 2)}{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}} \text{is undefined,} \qquad 1 \neq 2$$

3 The Identity Matrix, I

The identity matrix, denoted as I, is a special type of square matrix where all diagonal elements, elements running from the upper left to the lower right, are 1, and all other elements are 0. For example, the 2×2 identity matrix is:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When a square matrix A is multiplied by the identity matrix (provided the multiplication is defined), the result is the matrix A itself:

$$AI = A \quad \text{and} \quad IA = A$$

This property holds for matrices of any dimension as long as A is square.

4 The Determinant of a Matrix

The determinant of a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is calculated as

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

The determinant provides critical information about whether a matrix has an inverse. For a matrix A, the inverse exists if and only if $\det(A) \neq 0$. If $\det(A) = 0$, the matrix is called *singular* and does not have an inverse.

5 The Inverse of a Matrix

In arithmetic, to undo multiplication, we multiply by the multiplicative inverse of x, which is $\frac{1}{x}$. As long as $x \neq 0$, $x(\frac{1}{x}) = 1$. To undo matrix multiplication, we multiply by the matrix inverse, denoted as A^{-1} [1]. Just as multiplying a number by the reciprocal results in the identity element, 1, multiplying a matrix by its inverse returns the identity matrix, I.

Theorem: The inverse of a square matrix A, is denoted A^{-1}, if it exists, and has the property:

$$AA^{-1} = I \qquad \text{or} \qquad A^{-1}A = I$$

The inverse of a non-square matrix does not exist[1].

5.1 Calculating the Inverse of a 2×2 Matrix

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, if $\det(A) \neq 0$, the inverse of A can be calculated as:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

Now show that for a 2×2 matrix $AA^{-1} = I$. Again,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The inverse of A, provided $\det(A) \neq 0$, is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

$$\text{where } \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Now calculate AA^{-1}:

$$AA^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Rearrange and conduct the matrix multiplication:

$$AA^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$AA^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & -a_{11}a_{12} + a_{11}a_{12} \\ a_{21}a_{22} - a_{21}a_{22} & -a_{12}a_{21} + a_{11}a_{22} \end{bmatrix}$$

Simplify:

$$AA^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \det(A) & 0 \\ 0 & \det(A) \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, $AA^{-1} = I$.

Problem M.2.6: Determinant and Inverse

Given the matrix

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

Find the inverse of A, if it exists.

1. Calculate the determinant of A.

$$\det(A) = (4 \cdot 6) - (7 \cdot 2)$$
$$\det(A) = 24 - 14$$
$$\det(A) = 10$$

2. Determine if A has an inverse by examining the determinant.

Since $\det(A) = 10 \neq 0$, the matrix A has an inverse.

3. Calculate A^{-1}.
Using the formula for the inverse of a 2×2 matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Substitute the values from $A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$ and $\det(A) = 10$:

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

Now, distribute $\frac{1}{10}$ to each element:

$$A^{-1} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

Thus, the inverse of A is

$$\boxed{A^{-1} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}}$$

Problem M.2.7: Identity Matrix

Using the result of problem 5.1, show that $AA^{-1} = I$ and $A^{-1}A = I$.

First, show that $AA^{-1} = I$.

$$AA^{-1} = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 4 \cdot 0.6 + 7 \cdot -0.2 & 4 \cdot -0.7 + 7 \cdot 0.4 \\ 2 \cdot 0.6 + 6 \cdot -0.2 & 2 \cdot -0.7 + 6 \cdot 0.4 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 2.4 + -1.4 & -2.8 + 2.8 \\ 1.2 + -1.2 & -1.4 + 2.4 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now show that $A^{-1}A = I$.

$$A^{-1}A = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 0.6 \cdot 4 + -0.7 \cdot 2 & 0.6 \cdot 7 + -0.7 \cdot 6 \\ -0.2 \cdot 4 + 0.4 \cdot 2 & -0.2 \cdot 7 + 0.4 \cdot 6 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 2.4 + -1.4 & 4.3 + -4.3 \\ -0.8 + 0.8 & -1.4 + 2.4 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

6 Systems of Equations

Systems of linear equations can be solved efficiently using matrix methods, especially for larger systems. We will transform a system of linear equations into matrix-vector form, interpret the determinant of a matrix to understand the nature of solutions, and solve a two-variable system graphically and using the inverse of a matrix. These methods are essential for determining solutions to linear systems in both theoretical and practical applications[1].

A system of linear equations can be represented in matrix-vector form as:

$$A\vec{x} = \vec{b}$$

where A is the matrix of coefficients, $\vec{x}$ is the column vector of variables, and $\vec{b}$ is the solution vector. For example, the system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

can be rewritten in **matrix-vector form** as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Determinant and Solutions: The determinant of the coefficient matrix A (denoted $\det(A)$) provides insight into the nature of the solutions:

- If $\det(A) \neq 0$, the system has a unique solution.
- If $\det(A) = 0$, the system has either infinitely many solutions or no solution.

6.1 Solving Systems Graphically

Graphing each equation in the system reveals points of intersection, representing solutions where both equations are satisfied. For a system with a unique solution, the graphs intersect at a single point.

Consider the system:

$$\begin{cases} 2x_1 + x_2 = 5 \\ x_1 - x_2 = 1 \end{cases}$$

Each of theses lines are plotted in Figure 1.

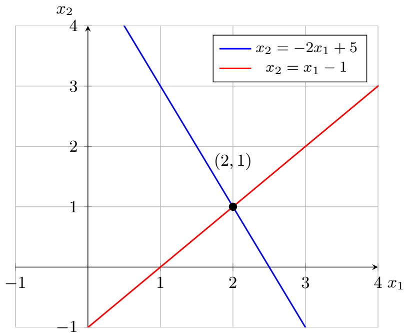


Figure 1: Plot of the system of equations. The point of intersection $(2, 1)$ is marked.

From the graph we can see the point of intersection at $(2, 1)$, so the solution is $x_1 = 2$ and $x_2 = 1$.

For graphical methods, if the lines are plotted one on top of the other, they are colinear and there are infinitely many solutions. If the lines are parallel, and not colinear, they do not intersect. Therefore, there are no solutions to the system of equations.

Problem M.2.8: Graphical Method

Solve the following system graphically.

$$\begin{cases} 2x_1 + 3x_2 = 13 \\ 4x_1 + x_2 = 11 \end{cases}$$

To solve this system graphically, we will first rewrite each equation in terms of x_1 and then plot the equations on a coordinate plane.

Step 1: Rewrite each equation in terms of x_1

First Equation: $2x_1 + 3x_2 = 13$

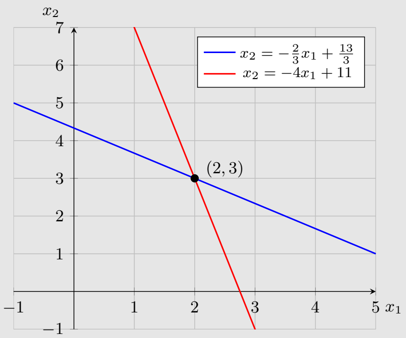
$$x_2 = -\frac{2}{3}x_1 + \frac{13}{3}$$

Second Equation: $4x_1 + x_2 = 11$

$$x_2 = -4x_1 + 11$$

Step 2: Plot each line.

We plot both equations and look for the point of intersection, which will represent the solution to the system.



Step 3: Identify the intersection point.
The lines intersect at the point $(2, 3)$, so the solution to the system is:

$$x_1 = 2, \quad x_2 = 3$$

6.2 Solving Using the Inverse Matrix

When $\det(A) \neq 0$, the inverse of A, denoted A^{-1}, can be used to solve for $\vec{x}$ as follows:

$$A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Problem M.2.9: Matrix Inverse Method

Solve the following system using matrix inverse method.

$$\begin{cases} 2x_1 + 3x_2 = 13 \\ 4x_1 + x_2 = 11 \end{cases}$$

Step 1: Write the System in Matrix Form

The system

$$\begin{cases} 2x_1 + 3x_2 = 13 \\ 4x_1 + x_2 = 11 \end{cases}$$

can be written as

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 11 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 13 \\ 11 \end{bmatrix}$.

Step 2: Calculate the Determinant of A

$$\det(A) = (2)(1) - (3)(4) = 2 - 12 = -10$$

Since $\det(A) \neq 0$, we know A has an inverse and that the system of equations has a unique solution. Therefore, we can solve for $\vec{x} = A^{-1}\vec{b}$.

Step 3: Find the Inverse of A

Using the formula for the inverse of a 2×2 matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we substitute $a = 2$, $b = 3$, $c = 4$, $d = 1$, and $\det(A) = -10$:

$$A^{-1} = \frac{1}{-10} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{bmatrix}$$

Step 4: Solve for $\vec{x}$
Now, calculate $\vec{x} = A^{-1}\vec{b}$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{bmatrix} \begin{bmatrix} 13 \\ 11 \end{bmatrix}$$

Calculate each entry:

$$x_1 = (-0.1)(13) + (0.3)(11) = -1.3 + 3.3 = 2$$

$$x_2 = (0.4)(13) + (-0.2)(11) = 5.2 - 2.2 = 3$$

Thus, the solution is:

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \text{or} \quad x_1 = 2, \quad x_2 = 3$$

References

[1] US Military Academy. *Modeling in a Real and Complex World*. West Point, New York: Department of Mathematical Sciences, 2022.