MA103: Mathematical Modeling

- Understand how vectors describe magnitude and direction
- Be familiar with vector notation and terminology
- Perform vector addition/subtraction and scalar multiplication
- Compute and interpret dot products
- Use vectors to model physical and geometric systems

Vectors

On the grid below, draw the position vectors $\vec{a}$ and $\vec{b}$ for $A(-3,1)$ and $B(2,3)$ respectively.

Solution:

$$\vec{a} = \langle -3, 1 \rangle, \quad \vec{b} = \langle 2, 3 \rangle$$

The displacement vector from A to B is:

$$\overrightarrow{AB} = \vec{b} - \vec{a} = \langle 2 - (-3), 3 - 1 \rangle = \langle 5, 2 \rangle$$

Thus, $\overrightarrow{AB} = \langle 5, 2 \rangle$ which is the same as $\vec{d} = \langle 5, 2 \rangle$.

Vector Operations:

Scalar Multiplication:
Given $c = 2$ and $\vec{s} = \langle 2, 1 \rangle$,

$$c\vec{s} = 2\langle 2, 1 \rangle = \langle 4, 2 \rangle$$

The new vector is twice as long in the same direction.

Vector Addition and Subtraction:
Given $\vec{u} = \langle 4, 2 \rangle$ and $\vec{v} = \langle -1, 3 \rangle$:

$$\vec{u} + \vec{v} = \langle 4 + (-1), 2 + 3 \rangle = \langle 3, 5 \rangle$$

$$\vec{v} - \vec{u} = \langle -1 - 4, 3 - 2 \rangle = \langle -5, 1 \rangle$$

Interpretation: $-\vec{u}$ represents a vector in the opposite direction of $\vec{u}$.

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1. **Magnitude of a Vector:**

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

It measures the vector's length.

2. Given $|\vec{w}| = 13$ and $\theta = 22.62^\circ$:

$$\vec{w} = \langle |\vec{w}| \cos\theta, |\vec{w}| \sin\theta \rangle = \langle 13\cos(22.62^\circ), 13\sin(22.62^\circ) \rangle = \langle 12, 5 \rangle$$

3. **Unit Vector:** A vector of magnitude 1 in the same direction as $\vec{s}$.

$$\hat{s} = \frac{\vec{s}}{|\vec{s}|} = \frac{\langle 3, 4 \rangle}{\sqrt{3^2 + 4^2}} = \frac{\langle 3, 4 \rangle}{5} = \langle 0.6, 0.8 \rangle$$

4. **Standard Basis Vectors:**

$$\hat{i} = \langle 1, 0, 0 \rangle, \quad \hat{j} = \langle 0, 1, 0 \rangle, \quad \hat{k} = \langle 0, 0, 1 \rangle$$

$$\vec{r} = \langle 4, 2, -7 \rangle = 4\hat{i} + 2\hat{j} - 7\hat{k}$$

Dot Product:

Given $\vec{u} = \langle -2, 4, 0 \rangle$, $\vec{v} = \langle \frac{1}{2}, 5, -2 \rangle$, $\vec{w} = \langle 1, -1, 2 \rangle$, and $\vec{x} = \langle -4, 6 \rangle$:

$$\vec{u} \cdot \vec{v} = (-2)(\tfrac{1}{2}) + (4)(5) + (0)(-2) = -1 + 20 = 19$$

$$\vec{u} \cdot \vec{w} = (-2)(1) + (4)(-1) + (0)(2) = -2 - 4 = -6$$

$$\vec{v} \cdot \vec{x} = \text{Not defined (different dimensions)}$$

$$\vec{w} \cdot \vec{x} = \text{Not defined (different dimensions)}$$

Problem 1
Solutions:

a) $\vec{F}_1 + \vec{F}_2 = \langle 4 + (-2), 3 + 6 \rangle = \langle 2, 9 \rangle$
Resultant Force: $\vec{F}_r = \langle 2, 9 \rangle$

b) $\overrightarrow{AB} = \langle 8 - 2, 9 - 5 \rangle = \langle 6, 4 \rangle$

c) $\vec{v}_{\text{total}} = \vec{v}_1 + \vec{v}_2 = \langle 200 + 20, 100 - 30, 50 + 10 \rangle = \langle 220, 70, 60 \rangle$ km/h

d) $5\vec{F} = 5\langle 3, -4 \rangle = \langle 15, -20 \rangle$

e) $2\vec{v} = 2\langle 2, 3, 4 \rangle = \langle 4, 6, 8 \rangle$

f) $|\vec{v}| = \sqrt{7^2 + (-24)^2} = \sqrt{49 + 576} = \sqrt{625} = 25$

g) $|\vec{v}| = \sqrt{10^2 + 20^2 + 30^2} = \sqrt{100 + 400 + 900} = \sqrt{1400} \approx 37.42$

h) $\tan\theta = \dfrac{12}{-5} = -2.4 \Rightarrow \theta = 180^\circ - \tan^{-1}(2.4) \approx 180^\circ - 67.38^\circ = 112.62^\circ$

i) $\vec{v}_r = \langle 3 + 1, 4 + (-2) \rangle = \langle 4, 2 \rangle$
$|\vec{v}_r| = \sqrt{4^2 + 2^2} = \sqrt{20} = 4.47$
$\theta = \tan^{-1}\left(\frac{2}{4}\right) = 26.57^\circ$

j) $\vec{d} = \langle 50\cos 30^\circ, 50\sin 30^\circ \rangle = \langle 43.3, 25 \rangle$ km