**Admin Notes / Agenda**

- Warm Up
- Lecture - inverse / transpose
- Zero Point Quiz and reveiw
- Schedule Review
- Board sheets

# Warm Up: Matrix Multiplication and Inverse

We will explore how to multiply a $2 \times 3$ matrix by a $3 \times 2$ matrix and then find the inverse of the resulting $2 \times 2$ matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

## Step 1: Multiply $A$ and $B$

Each entry in $AB$ is found by taking the dot product of a row of $A$ with a column of $B$.

$$AB = \begin{bmatrix} (1)(1) + (2)(0) + (3)(1) & (1)(0) + (2)(1) + (3)(2) \\ (4)(1) + (5)(0) + (6)(1) & (4)(0) + (5)(1) + (6)(2) \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 8 \\ 10 & 17 \end{bmatrix}$$

## Step 2: Find the Inverse of the Resulting $2 \times 2$ Matrix

Let

$$C = AB = \begin{bmatrix} 4 & 8 \\ 10 & 17 \end{bmatrix}.$$

Compute its determinant:

$$\det(C) = (4)(17) - (8)(10) = 68 - 80 = -12$$

Since $\det(C) \neq 0$, $C$ is invertible. The formula for the inverse of a $2 \times 2$ matrix is:

$$C^{-1} = \frac{1}{\det(C)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Substitute the values:

$$C^{-1} = \frac{1}{-12} \begin{bmatrix} 17 & -8 \\ -10 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{17}{12} & \frac{2}{3} \\ \frac{5}{6} & -\frac{1}{3} \end{bmatrix}$$

## Final Results

$$AB = \begin{bmatrix} 4 & 8 \\ 10 & 17 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} -\frac{17}{12} & \frac{2}{3} \\ \frac{5}{6} & -\frac{1}{3} \end{bmatrix}$$

**Summary:** - Multiplying a $2 \times 3$ by a $3 \times 2$ matrix gives a $2 \times 2$ result. - The resulting matrix is invertible because its determinant is nonzero.

# Summary of Key Terms and Operations

- **Matrix:** A matrix is a rectangular array of numbers arranged in rows and columns. A matrix with $m$ rows and $n$ columns is called an $m \times n$ *matrix*.
- **Matrix Notation:** A matrix is labeled with a capital letter (e.g., $A$, $B$), and its elements with subscripts. For example:

  $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

  where $a_{ij}$ is the entry in the $i$th row and $j$th column.
- **Rows and Columns:**
  - A *row* is a horizontal selection of elements in a matrix.
  - A *column* is a vertical selection of elements in a matrix.
- **Vectors as Matrices:** A vector in component form can be written as a matrix:
  - A *row vector* is a $1 \times n$ matrix, e.g. $\begin{bmatrix} 2 & -1 & 5 \end{bmatrix}$
  - A *column vector* is an $n \times 1$ matrix, e.g. $\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

  Both forms represent the same vector, just written differently for different operations.
- **Matrix Addition:** Two matrices must have the same dimensions. Add corresponding entries:

  $$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$
- **Matrix Subtraction:** Subtract corresponding entries (same-sized matrices only):

  $$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}$$
- **Scalar Multiplication:** Multiply each entry of a matrix by a real number $c$:

  $$cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

## Matrix Multiplication

Matrix multiplication combines rows of the first matrix with columns of the second using the **dot product**.

**Step 1: Start with the dot product.**

$$\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2(4) + 3(1) = 11$$

A $1 \times 2$ row vector times a $2 \times 1$ column vector gives a single number (a scalar).

**Step 2: Extend to a $2 \times 2$ matrix multiplication.**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1)(2) + (2)(1) & (1)(0) + (2)(5) \\ (3)(2) + (4)(1) & (3)(0) + (4)(5) \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 20 \end{bmatrix}$$

Each entry is found by taking a row of $A$ dotted with a column of $B$.

**Step 3: A non-square example ($2 \times 3$ times $3 \times 2$).**

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1)(3) + (0)(2) + (2)(1) & (1)(1) + (0)(1) + (2)(0) \\ (-1)(3) + (3)(2) + (1)(1) & (-1)(1) + (3)(1) + (1)(0) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

The result is a $2 \times 2$ matrix.

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## Matrix Transpose

The **transpose** of a matrix, written $A^T$, is formed by swapping rows and columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Explanation: - The first row of $A$ becomes the first column of $A^T$. - The second row of $A$ becomes the second column of $A^T$.

$A$ is $2 \times 3$, so $A^T$ is $3 \times 2$.

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## Identity Matrix

The **identity matrix**, denoted $I$, acts like the number 1 for matrix multiplication:

$$AI = IA = A$$

For $2 \times 2$ matrices:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Similarly, for $3 \times 3$:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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## Inverse of a Matrix

The **inverse** of a matrix $A$, written $A^{-1}$, is defined (for square matrices only) so that:

$$AA^{-1} = A^{-1}A = I$$

Not all matrices have inverses — only those with a nonzero determinant.
For a $2 \times 2$ matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{(2)(3) - (1)(5)} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \times \frac{1}{1}$$

Verification:

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

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