Admin Notes / Agenda

- Download prior quizzes and exams NLT 30 NOV
- Warm Up
- PSL

1 Key Definitions for What If

Linear programming models rely on several fundamental assumptions that ensure the relationships between variables remain linear and mathematically tractable:

1. **binding:** A constraint is binding if the left-hand side and the right-hand side of the constraint are equal when the optimal values of the variables are substituted into the constraint.
2. **non-binding:** A constraint is non-binding if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint.
3. **Shadow Price:** A shadow price is how much the objective function value is increased if the right-hand side value of a binding constraint is increased by 1, as long as it remains a binding constraint.
4. **What if: Interpret slopes of the objective:** For the objective function $Z = c_1x_1 + c_2x_2$, each pair (c_1, c_2) defines a contour line whose slope is $-\frac{c_1}{c_2}$. Sensitivity analysis in two dimensions therefore reduces to determining the range of slopes for which the contour line still supports the feasible region at the current optimal vertex. The allowable range is the interval of slopes for which no adjacent vertex yields a better objective value, ensuring that the supporting line touches the feasible region only at the optimal point.

2 Some Review:

1. **Solving for an unknown component of a velocity vector:** A particle moves with velocity
$$\vec{v} = \langle 3,\ v_2 \rangle$$
and the speed is
$$\|\vec{v}\| = 10.$$
Find the unknown component v_2.

2. **Finding the angle between two velocity vectors:** Two objects move with velocities
$$\vec{u} = \langle 4,\ -2 \rangle, \qquad \vec{w} = \langle 1,\ 5 \rangle.$$
Calculate the angle between $\vec{u}$ and $\vec{w}$.

3. **Solving a 2D system using a matrix inverse:** Consider the system
$$\begin{cases} 2x + 3y = 7, \\ -4x + y = -5. \end{cases}$$
   (a) Write this system in the form $A\vec{x} = \vec{b}$.
   (b) Compute A^{-1} and use it to solve $\vec{x}$.
   (c) What is $A^{-1}\,A$.
   (d) Describe what this solution means from a geometric and optimization perspective.

4. **Finding a missing vector component from magnitude:** A velocity vector is given by
$$\vec{V} = \langle V_1,\ -6,\ 0 \rangle,$$
and it is known that the magnitude of the velocity is
$$\|\vec{V}\| = 10.$$
Determine the missing component V_1.

5. **Compute an angle between two velocity vectors:** Given
$$\vec{v}_1 = \langle -3,\ 4 \rangle, \qquad \vec{v}_2 = \langle 2,\ 1 \rangle,$$
find the angle between the two acceleration vectors.

6. **Matrix multiplication :** Compute the product
$$\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 6 & 8 & 3 \end{pmatrix}.$$
Write the resulting vector in simplified form.

7. **List the four assumptions of linear programming:** List and briefly describe the four main assumptions of linear programming: additivity, proportionality, divisibility, and certainty.

3 Feasible Region Analysis and Objective Function Exploration

1. **Draw the feasible region and identify all intersection points and the optimal point.** Using the given system of linear constraints, sketch the feasible region in the x_1–x_2 plane. Clearly mark every corner point (vertex) formed by intersections of constraint boundaries, and determine which vertex is optimal for the original objective function.

2. **Explain why we may limit our search for optimal points to the vertices of the feasible region.** Provide a brief justification, using the principles we have discussed, for why linear programs achieve their optimum at a vertex of the feasible region rather than in the interior.

3. **Identify a new objective function and justify your choice.** Propose an alternative objective function and explain the motivation behind selecting new coefficients. Describe how this new objective direction changes the orientation of objective contour lines and how it may alter the optimal solution.

4. **Determine the allowable range for the cost of local fishing licenses that preserves the original optimal solution, and interpret the result.** Returning to the original objective function, treat the coefficient associated with local fishing licenses as a variable parameter. Determine the interval of values for this coefficient such that the current optimal vertex remains optimal. Interpret the meaning of this allowable range in practical terms—what does it tell us about price flexibility before the optimal decision changes?

[11pt]article [margin=1in]geometry xcolor amsmath amssymb lmodern [T1]fontenc

Original problem

A company produces 2 types of lemon flavored drinks: Regular and Charged Lemonade. The number of Charged Lemonade drinks is limited by machine mixing capacity and is limited to 10 per hour. Your pantry also has limited space to store lemons for processing. The regular Lemonade requires 2 lemons per large drink and the Charged one requires 5 lemons. You can only fit 60 lemons at a time in the pantry. The lemonades are also served in special cups and you only have 18 cups total available per hour. Finally, you are limited by the amount of ice your ice machine can make. The regular Lemonade usually has 3 scoops of ice per large drink, while the Charged Lemonade is full of caffeine and chemicals which leaves less room for ice. The Charged Lemonade only requires 1 scoop of ice per drink. Your ice machine only makes enough ice to fill 44 scoops total every hour. The Regular Lemonade generates a profit of $2 and the Charged Lemonade only generates 1$ per drink.

Decision variables: x_1 = number of Regular Lemonades, x_2 = number of Charged Lemonades.

$$\max_{x_1,x_2} P = 2x_1 + 1x_2$$

subject to

$$\begin{aligned}
x_2 &\le 10 && \text{(machine capacity)}\\
2x_1 + 5x_2 &\le 60 && \text{(lemons)}\\
x_1 + x_2 &\le 18 && \text{(cups)}\\
3x_1 + x_2 &\le 44 && \text{(ice)}\\
x_1, x_2 &\ge 0.
\end{aligned}$$

The optimal solution found earlier is $\boxed{(x_1, x_2) = (13, 5)}$ with objective value $P = 31$. The following answers refer to that optimal solution.

Questions and answers

1. Which constraints are binding?
   Binding constraints at $(13, 5)$:
   - Cups: $x_1 + x_2 = 18$ since $13 + 5 = 18$.
   - Ice: $3x_1 + x_2 = 44$ since $3(13) + 5 = 44$.
2. Which constraints are non-binding?
   Non-binding constraints at $(13, 5)$:
   - Machine capacity: $x_2 \le 10$ (slack $= 10 - 5 = 5$).
   - Lemons: $2x_1 + 5x_2 \le 60$ (slack $= 60 - (2 \cdot 13 + 5 \cdot 5) = 60 - 51 = 9$).
   - Nonnegativity: $x_1, x_2 \ge 0$ (both strictly positive, so not binding).
3. What if machine capacity decreases? How much can machine capacity decrease before it becomes a binding constraint? For what range of machine capacity is the optimal solution still optimal?
   At the current optimum $x_2 = 5$. The machine capacity constraint is $x_2 \le b$ (originally $b = 10$). As long as the capacity satisfies $b \ge 5$, the point $(13, 5)$ remains feasible and (since the same basis of binding constraints remains) remains the optimal solution. If b is decreased strictly below 5, $(13, 5)$ becomes infeasible and the optimal solution will change.
   Therefore: the machine capacity can decrease from 10 down to $\boxed{5}$ before it becomes binding. The range of machine capacities for which the current solution stays optimal is $\boxed{b \ge 5}$.
4. If we change the available cups from 18 to 19, will the optimal solution change? By how much does the objective function change?
   Yes — increasing the cups to 19 relaxes the cup constraint. The new intersection of the two original binding constraints (ice and cups) becomes the candidate optimum: solve

$$\begin{aligned}
x_1 + x_2 &= 19,\\
3x_1 + x_2 &= 44.
\end{aligned}$$

Subtracting gives $2x_1 = 25 \Rightarrow x_1 = 12.5$, and then $x_2 = 6.5$. This point is feasible with the other constraints (machine cap and lemons remain satisfied). The new objective is

$$P = 2(12.5) + 6.5 = 25 + 6.5 = 31.5.$$

So the optimal solution changes to $(x_1, x_2) = (12.5, 6.5)$ and the objective increases from 31 to 31.5, i.e. an increase of $\boxed{0.5}$ dollars.

5. What is the shadow price of cups?
   The shadow price (dual value) of the cups resource is the marginal increase in optimal profit per one-unit increase in the number of cups, while the current basis of binding constraints remains unchanged. From the calculation above, increasing cups from 18 to 19 raised the objective by 0.5. Hence the shadow price is

$$\boxed{\text{0.5 dollars per extra cup}}.$$

   (Valid while the basis remains the same — i.e. for small changes around 18; the computed change used a one-unit increase and did not change which constraints were binding besides the cup constraint.)

6. If the number of lemons required to make regular lemonade increases to 3, will the optimal solution change? For what range of lemons is the optimal solution still optimal?

   In particular, for $a = 3$ the new optimal solution (re-solving the LP) is obtained at the intersection of the two constraints

$$\begin{aligned} 3x_1 + 5x_2 &= 60 && \text{(lemons with } a = 3) \\ 3x_1 + x_2 &= 44 && \text{(ice)} \end{aligned}$$

   Solving gives $x_2 = 4$ and $x_1 = 40/3 \approx 13.3333$, with objective

$$P = 2(40/3) + 4 = \frac{80}{3} + 4 = \frac{92}{3} \approx 30.6667.$$

   So the objective falls from 31 to 30.6667 and the optimal solution changes when a is raised to 3.

7. How much can the number of available lemons change before the constraint becomes binding?