Admin Notes / Agenda

- Send me message about project ASAP
- Warm Up

1 Key Definitions for What If

Linear programming models rely on several fundamental assumptions that ensure the relationships between variables remain linear and mathematically tractable:

1. **binding:** A constraint is binding if the left-hand side and the right-hand side of the constraint are equal when the optimal values of the variables are substituted into the constraint.
2. **non-binding:** A constraint is non-binding if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint.
3. **Shadow Price:** A shadow price is how much the objective function value is increased if the right-hand side value of a binding constraint is increased by 1, as long as it remains a binding constraint.
4. **What if: Interpret slopes of the objective:** For the objective function $Z = c_1x_1 + c_2x_2$, each pair (c_1, c_2) defines a contour line whose slope is $-\frac{c_1}{c_2}$. Sensitivity analysis in two dimensions therefore reduces to determining the range of slopes for which the contour line still supports the feasible region at the current optimal vertex. The allowable range is the interval of slopes for which no adjacent vertex yields a better objective value, ensuring that the supporting line touches the feasible region only at the optimal point.

Key Concepts, Vocabulary, and Clauses for Robust Optimization

1. The Lemonade Linear Program

A company produces two types of lemon drinks: *Regular Lemonade* (x_1) and *Charged Lemonade* (x_2). The profit-maximization LP is:

$$\max_{x_1,x_2} P = 2x_1 + x_2$$

subject to resource constraints:

$$
\begin{aligned}
x_2 &\le 10 && \text{(Mixing capacity)} \\
2x_1 + 5x_2 &\le 60 && \text{(Pantry lemon limit)} \\
x_1 + x_2 &\le 18 && \text{(Cups available)} \\
3x_1 + x_2 &\le 44 && \text{(Ice scoops limit)} \\
x_1, x_2 &\ge 0
\end{aligned}
$$

2. Core Definitions and Vocabulary

- **What-if analysis**: Evaluates how changes in parameters affect feasibility or optimality.
- **Technological coefficients**: Coefficients such as 2, 5, 3, and 1 in the constraints; represent resources needed per drink.
- **Binding constraint**: A constraint that holds with equality at the optimal solution.
- **Robust solution**: A solution that remains feasible across all plausible parameter values.
- **Infeasibility under uncertainty**: When small parameter changes make the original optimal solution unusable.

3. Robust Optimization Framework

- **Robust optimization** selects values of x_1, x_2 that remain feasible in the *worst-case scenario* for uncertain parameters.
- Under the **additivity assumption** of LPs, parameters can be assigned worst-case values *independently*.
- **Conservative or Worst-case logic**:
  - For constraints, worst-case = larger technological coefficients or smaller right-hand sides (tightens feasibility).
  - For maximization objective coefficients, worst-case = minimum profit values.

4. Example of Parameter Uncertainty (Lemonade Context)

Suppose the following uncertainties exist:

- The mixing capacity for Charged Lemonade x_2 may range from 8 to 10 per hour.
- The lemons needed per drink may vary:

$$a_{11} \in [1.5, 2.5], \quad a_{12} \in [4, 6]$$

- Cup availability may drop from 18 to somewhere in the range $[14, 18]$.
- Ice machine output may drop to as low as 36 scoops per hour.

Then the **worst-case parameters** would be:

- Mixing limit: $x_2 \le 8$
- Lemon constraint: $2.5x_1 + 6x_2 \le 60$
- Cups: $x_1 + x_2 \le 14$
- Ice scoops: $3x_1 + x_2 \le 36$

5. Worst-Case Robust Constraint Set

The robustified LP becomes:

$$\max_{x_1,x_2} \; P = 2x_1 + x_2$$

$$\text{s.t.}$$

$$\begin{aligned} x_2 &\le 8 \\ 2.5x_1 + 6x_2 &\le 60 \\ x_1 + x_2 &\le 14 \\ 3x_1 + x_2 &\le 36 \\ x_1, x_2 &\ge 0 \end{aligned}$$

The key robust constraint is the most restrictive one after applying worst-case values.

6. Interpreting the Robust Solution

- The robust solution remains feasible for *all* parameter values in the uncertainty ranges.
- It will not be the optimal solution for the best-case scenario, but it guarantees feasibility even under reduced mixing capacity, fewer cups, less ice, or higher lemon usage.
- This reflects the tradeoff between **optimality** (max profit) and **reliability** (always feasible).

7. Key Evaluation Questions

- What risks or uncertainties does the robust solution protect against?
- How far is it from the optimal non-robust solution?
- What happens if the non-robust solution becomes infeasible?
- Are the uncertainty ranges supported by data or expertise?
- Are the tradeoffs clearly communicated to decision-makers?

8. Ethical Considerations

- **Data validity:** Are uncertainty bounds evidence-based?
- **Model validity:** Who bears the cost of being overly conservative or overly optimistic?
- **Stakeholder impact:** How does feasibility failure affect operations?
- **Communication:** Have the implications of robust choices been made clear?